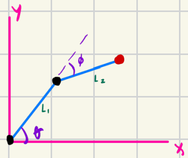
Inverse Kinematics

What's inverse kinematics (IK)?

- IK is a crucial concept in robotics. It is the study of finding limb angles given a target point for the end effector (the end of the arm)
- It's called <u>inverse</u> kinematics because it's precicely that... backwards kinematics
- You might have studied kinematics in your physics class, but basically with kinematics, you are given the angles of the arm and determine what position the end effector will be.

ex:

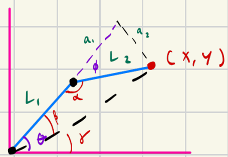


using basic trig, we can see that the coordinates of the end effector will be

$$\left(L_1\cos\theta + L_2\cos(\theta-\phi),\ L_1\sin\theta + L_2\sin(\theta-\phi)\right)$$

- Now for inverse kinematics

ex 2:



Using geometry, we can use law of cosines to solve as follows:

$$\cos\alpha = \frac{L_1^2+L_2^2-(x^2+y^2)}{2L_1L_2}\ ;\quad \alpha = \cos^{-1}\left(\frac{L_1^2+L_2^2-(x^2+y^2)}{2L_1L_2}\right)\ ;\quad \underline{\phi = 180-\alpha\ ;\quad \cos\alpha = -\cos\phi}$$

$$a_1 = L_2\cos\phi\ ,\quad a_2 = L_2\sin\phi$$

$$x^2+y^2 = (L_1+L_2\cos\phi)\cos\theta\ ;\quad \beta = \cos^{-1}\left(\frac{\sqrt{x^2+y^2}}{L_1+L_2\cos\phi}\right)$$

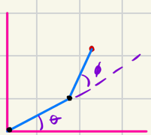
$$\tan\gamma = \frac{y}{x}\ ;\quad \gamma = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\underline{\theta = \beta+\gamma}$$

<u>☆ step 5 uses this</u>

Note that in this example, different angles for θ and ϕ can be made if the orientation of the "elbow" was flipped:

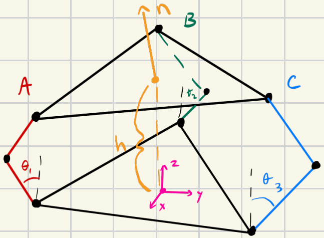
ex 3:



But the general process for solving is quite similar

IK can be much more complex depending on the amount of limbs of the legs or if extended in the third dimension. This is the simplest example and all we'll need for solving for our robot.

Steps to Solve 3RRS IK Problem



Input: θ (rotation for y axis), ϕ (rotation for x axis), h (height from origin)

1) Calculate top-plate triangle pts, calculate centroid
2) Calculate rotation matrix
3) Multiply z-vals of triangle pts ONLY with rotation matrix
4) Calculate translation vector so new centroid = old centroid
5) Use law of cosines to solve 2D IK for each leg using top plate triangle pts and bottom plate triangle pts

Output: $\theta_1, \theta_2, \theta_3$, the angles of each leg from the base triangle

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Step 0: Establish reference frame and constants

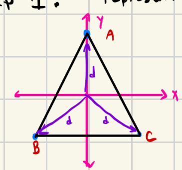
Origin: center of base triangle, leg A lies on y axis (YZ plane)

d: distance from joint to center

e: length of lower limbs

f: length of upper limbs

Step 1: represent $V_{A_i}, V_{B_i}, V_{C_i}$ as initial coords of the triangle



$$V_{A_i} = \begin{bmatrix} 0 \\ d \\ h \end{bmatrix} \qquad V_{B_i} = \begin{bmatrix} -\frac{d\sqrt{3}}{2} \\ -\frac{d}{2} \\ h \end{bmatrix} \qquad V_{C_i} = \begin{bmatrix} \frac{d\sqrt{3}}{2} \\ -\frac{d}{2} \\ h \end{bmatrix}$$

Calculate centroid:

$$\begin{bmatrix} (0 - \frac{d\sqrt{3}}{2} + \frac{d\sqrt{3}}{2})/3 \\ (d - \frac{d}{2} - \frac{d}{2})/3 \\ (h \quad h \quad h)/3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ h \end{bmatrix}$$

Step 2: Rotation Matrix

Since we can only rotate about the x and y axes, combining an x and y rotation matrix will give us a combined rotation matrix to apply to our triangle

$$R = R_y R_x = \begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

These matrices can be derived (or searched up online!!)

Step 3: Rotate triangle!

Normally, we would multiply our rotation matrix to each point, but we choose to only change z-vals! I encourage you to think about why that may be !! :)

Let's call our changed pts VA_R, VB_R, VC_R, where x and y vals of $VA_R, VB_R, VC_R = VA_I, VB_I, VC_I$

This creates a slight distortion, and our equillateral triangle isn't really equillateral anymore... but for small angle changes, this distortion is small enough to not be significant

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Step 4: Rotating our triangle also changed the height of the middle of the triangle (centroid). Unless we want the height of platform to keep changing for each calculation, we should translate the points so the centroids will be equal.

Let's call T our translation vector, where $T = \text{Centroid}_R - \text{Centroid}_I$

Now add T to VA_R, VB_R, VC_R. These are now our final rotated pts, VA_f, VB_f, VC_f

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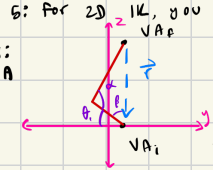
Step 5: For 2D IK, you can project a vector from the base to each pt to solve

ex: Leg A



$$V_{A_B} = V_{A_i}$$

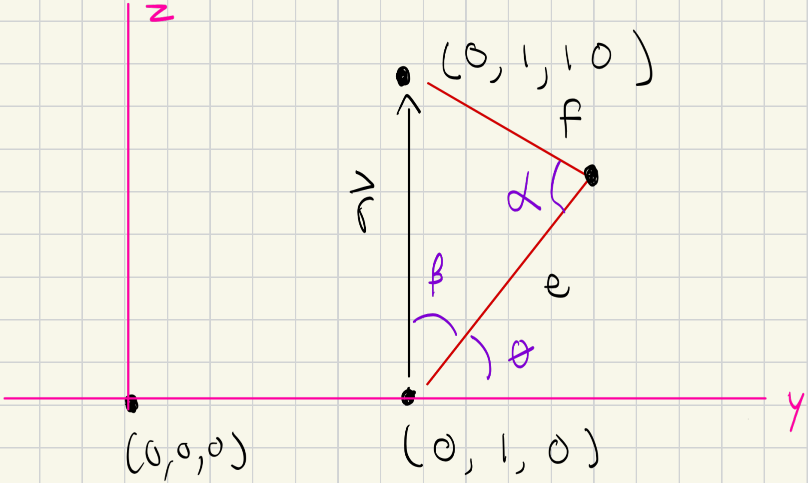
$$\vec{r} = V_{A_f} - V_{A_B}$$

- use * eqs from page 1 to find θ_1 and repeat for θ_2, θ_3

We always want the "elbows" to be outside, so $\theta_1, \theta_2, \theta_3$ must be acute

Since only z vals are changing, $\vec{r}$ will always just end up being purely vertical. This trivializes calculations for angles, and uses the same formula for the other legs! (This is shown in the next page)

Examples for Step 5

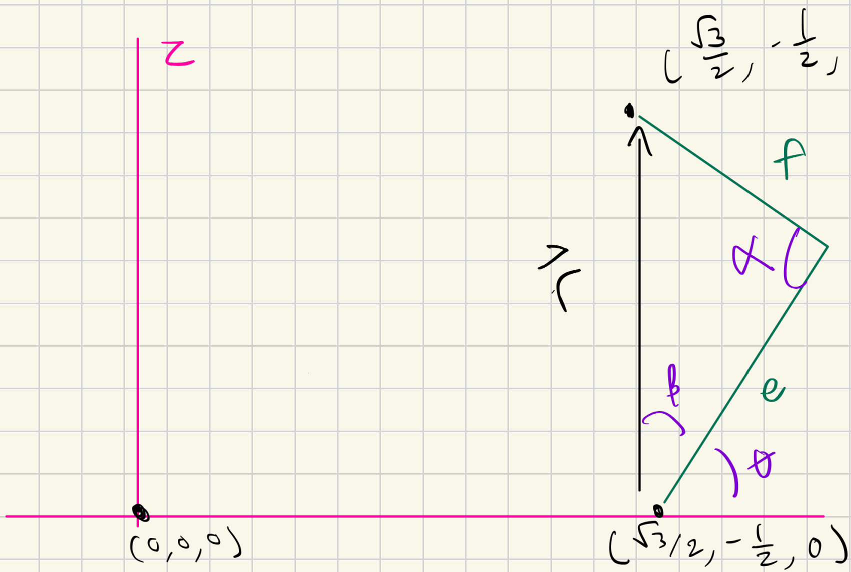


$r = (0, 0, 10)$

$\theta = 90 - \beta$

$\alpha = acos(\ldots)$

$\beta = acos(\ldots)$



$\vec{r} = (0, 0, 10)$

$\theta = 90 - \beta$